Lesson 17 (3.9)

Today: * Derivatives of Logarithmic functions

* Logarithmic Differentiation

Office Hours: MWF: 2:45PM – 4:15PM, MATH 842.

Announcements:

* Exam 2: Wednesday, Oct 15th 8PM – 9PM
* Final Exam: Monday, Dec 15th 1PM – 3PM

} Instructions on Brightspace

* HW 16, 17 : Due on Tuesday
* Quiz 10 (Lesson 15) : On Tuesday

Review: Properties of Logarithms

$b > 0$

* $x = e^y$ then $y = \ln x$

  $x = b^y$ then $y = \log_b x$

* $\log_b mn = \log_b m + \log_b n$

* $\log_b m/n = \log_b m - \log_b n$

* $\log_b m^r = r \log_b m$

* $b^{\log_b x} = x$

* $\log_b b^x = x$

if $b = e$

$e^{\ln x} = \ln e^x = x.$

Derivative of $y = \ln x\,, \quad x > 0$

$$y' = \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h}$$

$\vdots$

Not easy to compute

$$\boxed{\frac{d}{dx} \ln x = \frac{1}{x}\,, \quad x > 0}$$

$x = e^y$

implicit Differentiation

① $\frac{d}{dx}$ on Both sides

$$\frac{d}{dx} x = \frac{d}{dx} e^y$$

$$1 = \frac{d}{dy} e^y \cdot \frac{dy}{dx}$$

$$1 = e^y \cdot y'$$

② Solve for y'

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

$$y = \ln|x| = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

when $x < 0$ $\rightsquigarrow$ find $\frac{d}{dx}\left(\ln(-x)\right)$

$u = -x > 0$

Chain Rule

$u > 0$

$$= \frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \frac{d}{dx}(-x)$$

$$= \frac{1}{-x} \cdot -1 = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0}$$

eg: $y = \ln(e^x)$

Find $\frac{dy}{dx}$ by using Chain Rule

$y = \ln e^x = x$

$y' = 1$

$$\frac{dy}{dx} = \frac{d}{dx}\ln(\overset{u}{e^x}) = \frac{d}{du}\ln u \cdot \frac{dy}{dx}$$

$$= \frac{1}{u} \cdot \frac{d}{dx}e^x$$

$$= \frac{1}{e^x} \cdot e^x$$

$$= 1$$

eg: $y = \ln(\sin^4 x)$, find y'

$= 4\ln(\sin x)$.

$$\frac{d}{dx}y = 4 \cdot \frac{d}{dx}\ln(\overset{u}{\sin x})$$

$$= 4\frac{d}{dx}\ln u = 4\left(\frac{d}{du}\ln u \cdot \frac{du}{dx}\right)$$

$$= 4\left[\frac{1}{u} \cdot \frac{d\sin x}{dx}\right] = 4\left[\frac{1}{\sin x} \cdot \cos x\right]$$

$$= 4\cot x .$$

eg: $y = \ln\left[\dfrac{x^4-3x^2+1}{(x+5)(x-3)}\right]$, find y'

$\ln(m/n) = \ln m - \ln n$

$y = \ln(x^4-3x^2+1) - \left[\ln\left((x+5)(x-3)\right)\right]$

$= \ln(x^4-3x^2+1) - \left[\ln(x+5) + \ln(x-3)\right]$

$= \ln(x^4-3x^2+1) - \ln(x+5) - \ln(x-3)$

$y' = \dfrac{d}{dx}\ln(x^4-3x^2+1) - \dfrac{d}{dx}\ln(x+5) - \dfrac{d}{dx}\ln(x-3)$

$= \dfrac{1}{x^4-3x^2+1}\cdot\underbrace{\dfrac{d}{dx}(x^4-3x^2+1)}_{4x^3-6x^2} - \dfrac{1}{x+5}\cdot\underbrace{\dfrac{d}{dx}(x+5)}_{1} - \dfrac{1}{x-3}\underbrace{\dfrac{d}{dx}(x-3)}_{1}$

$= \dfrac{4x^3-6x}{x^4-3x^2+1} - \dfrac{1}{x+5} - \dfrac{1}{x-3}.$

$$\frac{d}{dx}(x^7) = 7x^6$$

Power Rule Applies only when Power is a number NOT Function.

$$\frac{d}{dx}(e^7) = 0$$

$$\frac{d}{dx}(7^e) = 0$$

Make use of logarithm to compute Derivative.

$$\frac{d}{dx}(7^x) \neq x\,7^{x-1}$$

$$\frac{d}{dx}(x^{2x}) \neq (2x)\,x^{2x-1}$$

Logarithmic Differentiation

$$y = [f(x)]^{g(x)}$$

eg: x^{2x}

$x^{\sin x}$

7^{x}

① take ln on both sides

$$\ln y = \ln\left[(f(x))^{g(x)}\right] = g(x) \cdot \ln(f(x))$$

② take $\frac{d}{dx}$ on Both sides

$$\underbrace{\frac{d}{dx}\ln y}_{\text{Chain Rule}} = \underbrace{\frac{d}{dx}\left[g(x) \cdot \ln(f(x))\right]}_{\text{Apply Product Rule.}}$$

$\frac{1}{y} \cdot y'$

③ Multiply with y on Both Sides.

eg! $y = x^{2x}$

① take ln on both

$\ln y = \ln(x^{2x}) = 2x \cdot \ln x.$

② $\frac{d}{dx}$ on Both sides

$$\underbrace{\frac{d}{dx}\ln y}_{\frac{1}{y}\cdot y'} = \underbrace{\frac{d}{dx}\left[2x\ln x\right]}_{\text{product Rul}}$$

$$2\cdot\ln x + 2x\cdot\frac{1}{x} = 2\ln x + 2$$

$$\frac{1}{y}y' = 2\ln x + 2$$

③ Multipy y on Both sides

$$y' = y(2\ln x + 2) \Rightarrow \boxed{y' = x^{2x}(2\ln x + 2)}$$

eg: $y = x^{\sin x}$

① take ln on Both sides

$$\ln y = \ln(x^{\sin x}) = \sin x \cdot \ln x$$

② $\frac{d}{dx}$ on Both sides

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\sin x \cdot \ln x\right]$$

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \frac{\sin x}{x}$$

③ Multiply with y on both sides

$$y' = y\left[\cos x \cdot \ln x + \frac{\sin x}{x}\right] \rightarrow \boxed{y' = x^{\sin x}\left[\cos x \ln x + \frac{\sin x}{x}\right]}$$

eg: $y = 7^x$

$\ln y = x \cdot \ln 7$

$\frac{d}{dx} \ln y = \frac{d}{dx}(x \cdot \ln 7)$

$\frac{1}{y} \cdot y' = \ln 7$

$\rightsquigarrow \; y' = y \ln 7$

$y' = 7^x \cdot \ln 7$

$b > 0$

$y = b^x$

$\ln y = x \cdot \ln b$

$\frac{d}{dx} \ln y = \frac{d}{dx}(x \cdot \ln b)$

$\frac{1}{y} \cdot y' = \ln b$

$\rightsquigarrow \; y' = y \ln b$

$y' = b^x \cdot \ln b$

$$\boxed{\frac{d}{dx} b^x = b^x \cdot \ln b \, , \quad b > 0}$$

eg: $y = \log_b x \quad , b > 0$

$x = b^y$

$\frac{d}{dx}[x] = \frac{d}{dx} b^y$

$1 = \frac{d}{dy} b^y \cdot \frac{dy}{dx}$

$\boxed{\frac{d}{dy} b^y = b^y \cdot \ln b}$

$1 = (b^y \ln b) \cdot y' \rightsquigarrow y' = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$

$$\boxed{\frac{d}{dx} \log_b x = \frac{1}{x \ln b}}$$